CORRECTED PROOF OF THEOREM 1 OF: A JOINT DATA COMPRESSION AND TIME-DELAY ESTIMATION METHOD FOR DISTRIBUTED SYSTEMS VIA EXTREMUM ENCODING

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1. PROOF OF THEOREM 1

In order to prove the theorem, we shall use the following lemmas, whose proofs are given below.

Lemma 1 *Let* $\mathsf{j} \triangleq \arg\max_{0\leq n\leq N-1} \mathsf{x}[n]$, *where* $\{\mathsf{x}[n]\}_{n=0}^{N-1}$ *are iid standard normal. Then, for any* $\tau \in \mathbb{R}$,

$$\mathbb{P}(\mathsf{x}[\mathsf{j}] < \tau) \leq e^{-2^k\left(\frac{\tau}{1+\tau^2}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{\tau^2}{2}}}. \tag{1}$$

Furthermore, if we choose $\tau_*(k) \triangleq \sqrt{2\log(2)k(1-\varepsilon(k))} = \sqrt{2\log(N)}(1+o(1))$, *where* $\varepsilon(k) \triangleq \frac{1}{\sqrt{k}} = o(1)$, *we obtain*

$$\mathbb{P}(\mathsf{x}[\mathsf{j}] < \tau_*(k)) \leq e^{-2^{\sqrt{k}}\cdot\frac{1}{\sqrt{2\pi}}\left(\frac{\sqrt{2\log(2)k(1-\varepsilon(k))}}{1+2\log(2)k(1-\varepsilon(k))}\right)} = o\left(2^{-k}\right). \tag{2}$$

Lemma 2 *Let* $\mathsf{v}, \mathsf{z} \sim \mathcal{N}(0,1)$ *be independent, and* $\mathsf{u} \triangleq \min(\mathsf{v}, V)$, *for some* $V \in \mathbb{R}$. *Then, for any* $a \in \mathbb{R}$,

$$\mathbb{P}(a < \rho\mathsf{u} + \bar{\rho}\mathsf{z}) \geq \mathbb{P}(a < \mathsf{v}) - Q(V). \tag{3}$$

Proof of Lemma 1: For $\tau > 0$, we have

$$\begin{aligned}
\mathbb{P}(\mathsf{x}[\mathsf{j}] < \tau) &= \mathbb{P}\Big(\max_{1\leq n\leq N} \mathsf{x}[n] < \tau\Big) && (4)\\
&= \mathbb{P}(\mathsf{x}[1] < \tau, \ldots, \mathsf{x}[N] < \tau) && (5)\\
&= \mathbb{P}(\mathsf{x}[1] < \tau)^N && (6)\\
&= (1 - Q(\tau))^N && (7)\\
&\leq \left(1 - \left(\frac{\tau}{1+\tau^2}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{\tau^2}{2}}\right)^N && (8)\\
&\leq e^{-N\left(\frac{\tau}{1+\tau^2}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{\tau^2}{2}}}, && (9)
\end{aligned}$$

where we have used:

- $Q(x) \geq \frac{x}{(1+x^2)}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ for $x > 0$ in (8) [1, Eq. (10)]; and
- $1 - x \leq e^{-x} \Rightarrow (1-x)^N \leq e^{-xN}$ in (9).

Choosing $\tau = \tau_*(k)$ gives, after simplifying, (2). ■

Proof of Lemma 2: We have,

$$\begin{aligned}
&\mathbb{P}(a < \rho \mathsf{u} + \bar{\rho}\mathsf{z}) && (10)\\
&= \mathbb{E}\left[\mathbb{P}(a < \rho \mathsf{u} + \bar{\rho}\mathsf{z} \mid \mathsf{z})\right] && (11)\\
&= \mathbb{E}\left[\mathbb{P}\left(\tfrac{a-\bar{\rho}\mathsf{z}}{\rho} < \mathsf{u} \mid \mathsf{z}\right)\right] && (12)\\
&= \mathbb{E}\left[\frac{1}{1-Q(V)} \int_{\frac{a-\bar{\rho}\mathsf{z}}{\rho}}^{V} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x\right] && (13)\\
&\geq \mathbb{E}\left[\int_{\frac{a-\bar{\rho}\mathsf{z}}{\rho}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x - \int_{V}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x\right] && (14)\\
&= \mathbb{E}\left[\mathbb{P}\left(\tfrac{a-\bar{\rho}\mathsf{z}}{\rho} < \mathsf{v} \mid \mathsf{z}\right)\right] - Q(V) && (15)\\
&= \mathbb{P}(a < \mathsf{v}) - Q(V), && (16)
\end{aligned}$$

where (14) is from $[1-Q(V)]^{-1} > 1$, and (16) follows from the fact that $\rho^2 + \bar{\rho}^2 = 1$ and that v and z are independent. ■

Proof of Theorem 1: We start by deriving the upper bound. For brevity in the following derivation, Let $\mathsf{v} \sim \mathcal{N}(0,1)$. Using this notation, we have,

$$\begin{aligned}
&\mathbb{P}\left(\widehat{\mathsf{d}} \neq \mathsf{d} \,\middle|\, \mathsf{d}, \mathsf{x}[\mathsf{j}]\right) && (17)\\
&= \mathbb{P}\left(\bigcup_{\substack{\ell \in \mathcal{D} \\ \ell \neq \mathsf{d}}} \widehat{\rho}(\mathsf{d}) < \widehat{\rho}(\ell) \,\middle|\, \mathsf{d}, \mathsf{x}[\mathsf{j}]\right) && (18)\\
&\leq 2d_m \mathbb{P}(\widehat{\rho}(\mathsf{d}) < \widehat{\rho}(\ell) | \, \mathsf{d}, \mathsf{x}[\mathsf{j}]) && (19)\\
&\leq 2d_m \mathbb{P}\left(\rho \mathsf{x}[\mathsf{j}] < \sqrt{2-\rho^2}\mathsf{v} \,\middle|\, \mathsf{d}, \mathsf{x}[\mathsf{j}]\right) && (20)\\
&= 2d_m Q\left(\frac{\rho \mathsf{x}[\mathsf{j}]}{\sqrt{2-\rho^2}}\right), && (21)
\end{aligned}$$

where:

- In (19), we have used the union bound; and
- In (20), replacing $\mathsf{y}[\mathsf{j}+\ell]$ by v can only increase the probability. To see this more clearly, we first recall that $\widehat{\rho}(\ell)$ is merely a scaled version of $\mathsf{y}[\mathsf{j}+\ell]$. Then, we observe that $\mathsf{y}[\mathsf{j}+\ell] = \rho\mathsf{x}[\mathsf{j}-\mathsf{d}+\ell] + \bar{\rho}\mathsf{z}[\mathsf{j}+\ell]$ is a convex combination of a (possibly) one-sided (upper bounded) truncated standard Gaussian RV ($\mathsf{x}[\mathsf{j}-\mathsf{d}+\ell]$) and a standard Gaussian RV ($\mathsf{z}[\mathsf{j}+\ell]$), which are independent. Since v can be thought of as a convex combination with the same coefficients of two independent standard Gaussian RVs, it is interpreted as replacing the truncated Gaussian $\mathsf{x}[\mathsf{j}-\mathsf{d}+\ell]$ with a standard Gaussian, which can only increase the probability that $\mathsf{y}[\mathsf{j}+\mathsf{d}] < \mathsf{y}[\mathsf{j}+\ell]$.

Now, using the law of total expectation, the conditional upper bound (21) and Lemma 1, we obtain,

$$\begin{aligned}
\mathbb{P}\left(\widehat{\mathsf{d}} \neq \mathsf{d}\right) &= \mathbb{E}\left[\mathbb{P}\left(\widehat{\mathsf{d}} \neq \mathsf{d} \,\middle|\, \mathsf{d}, \mathsf{x}[\mathsf{j}]\right)\right] && (22)\\
&\leq \mathbb{E}\left[2d_m Q\left(\frac{\rho \mathsf{x}[\mathsf{j}]}{\sqrt{2-\rho^2}}\right)\right] && (23)\\
&\leq 2d_m Q\left(\frac{\rho \tau_*(k)}{\sqrt{2-\rho^2}}\right) + 2d_m \mathbb{P}(\mathsf{x}[\mathsf{j}] < \tau_*(k)) && (24)\\
&= 2d_m Q\left(\rho\sqrt{\frac{2\log(N)}{2-\rho^2}}\right)(1+o(1)), && (25)
\end{aligned}$$

where we recall in particular (2), namely $\mathbb{P}(\mathsf{x}[\mathsf{j}] < \tau_*(k)) = o\left(2^{-k}\right)$.

For the lower bound, we have,

$$
\begin{aligned}
&\mathbb{P}\left(\widehat{\mathsf{d}} \neq \mathsf{d} \middle| \mathsf{d}, \mathsf{x}[\mathsf{j}]\right) && (26)\\
&= \mathbb{P}\left(\bigcup_{\substack{\ell\in\mathcal{D}\\ \ell\neq \mathsf{d}}} \widehat{\rho}(\mathsf{d}) < \widehat{\rho}(\ell) \middle| \mathsf{d}, \mathsf{x}[\mathsf{j}]\right) && (27)\\
&\geq \mathbb{P}(\widehat{\rho}(\mathsf{d}) < \widehat{\rho}(\ell) | \mathsf{d}, \mathsf{x}[\mathsf{j}]) && (28)\\
&= \mathbb{P}(\mathsf{y}[\mathsf{j}+\mathsf{d}] < \mathsf{y}[\mathsf{j}+\ell] | \mathsf{d}, \mathsf{x}[\mathsf{j}]) && (29)\\
&= \mathbb{E}\left[\mathbb{P}(\mathsf{y}[\mathsf{j}+\mathsf{d}] < \mathsf{y}[\mathsf{j}+\ell] | \mathsf{d}, \mathsf{x}[\mathsf{j}], \mathsf{z}[\mathsf{j}+\mathsf{d}]) | \mathsf{d}, \mathsf{x}[\mathsf{j}]\right] && (30)\\
&\geq \mathbb{E}\left[\mathbb{P}(\mathsf{y}[\mathsf{j}+\mathsf{d}] < \mathsf{v} | \mathsf{d}, \mathsf{x}[\mathsf{j}], \mathsf{z}[\mathsf{j}+\mathsf{d}]) - Q(\mathsf{x}[\mathsf{j}]) | \mathsf{d}, \mathsf{x}[\mathsf{j}]\right] && (31)\\
&= \mathbb{P}(\mathsf{y}[\mathsf{j}+\mathsf{d}] < \mathsf{v} | \mathsf{d}, \mathsf{x}[\mathsf{j}]) - Q(\mathsf{x}[\mathsf{j}]) && (32)\\
&= \mathbb{P}\left(\rho\mathsf{x}[\mathsf{j}] < \sqrt{2-\rho^2}\,\mathsf{v} \middle| \mathsf{d}, \mathsf{x}[\mathsf{j}]\right) - Q(\mathsf{x}[\mathsf{j}]) && (33)\\
&= Q\left(\frac{\rho\mathsf{x}[\mathsf{j}]}{\sqrt{2-\rho^2}}\right) - Q(\mathsf{x}[\mathsf{j}]) && (34)
\end{aligned}
$$

where in (28) we have taken only one event of the union of events, and in (31) we have used Lemma 2. Using the law of total expectation, the lower bound (34) and Lemma 1, we obtain,

$$
\begin{aligned}
\mathbb{P}\left(\widehat{\mathsf{d}} \neq \mathsf{d}\right) &= \mathbb{E}\left[\mathbb{P}\left(\widehat{\mathsf{d}} \neq \mathsf{d} \middle| \mathsf{d}, \mathsf{x}[\mathsf{j}]\right)\right] && (35)\\
&\geq \mathbb{E}\left[Q\left(\frac{\rho\mathsf{x}[\mathsf{j}]}{\sqrt{2-\rho^2}}\right) - Q(\mathsf{x}[\mathsf{j}])\right] && (36)\\
&\geq \mathbb{P}\left(\tau_*(k) < \mathsf{x}[\mathsf{j}] < \sqrt{2\log(N)}\right) \mathbb{E}\left[Q\left(\frac{\rho\mathsf{x}[\mathsf{j}]}{\sqrt{2-\rho^2}}\right) - Q(\mathsf{x}[\mathsf{j}]) \middle| \tau_*(k) < \mathsf{x}[\mathsf{j}] < \sqrt{2\log(N)}\right] && (37)\\
&= Q\left(\rho\sqrt{\frac{2\log(N)}{2-\rho^2}}\right)(1+o(1)), && (38)
\end{aligned}
$$

since $\mathbb{P}(\tau_*(k) < \mathsf{x}[\mathsf{j}]) = 1 - o(1)$ and $\mathbb{P}\left(\mathsf{x}[\mathsf{j}] < \sqrt{2\log(N)}\right) = 1 - o(1)$. ■

Remark 1 *The upper and lower bounds above are essentially exponentially equivalent to those given in the original paper. Here, they are in terms of the Q-function, while in the paper they are in the form of exponential functions. Of course, Corollary 2 of the paper (Asymptotic error exponent) can be obtained with the upper and lower bounds established here.*

2. REFERENCES

[1] R. D. Gordon, "Values of Mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument," *Ann. Math. Statist.*, vol. 12, no. 3, pp. 364–366, 1941.